

PT symmetry and Weyl asymptotics

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In memory of Leon Ehrenpreis

Abstract

For a class of PT-symmetric operators with small random perturbations, the eigenvalues obey Weyl asymptotics with probability close to 1. Consequently, when the principal symbol is non-real, there are many non-real eigenvalues.

Contents

1	Introduction	1
2	The semi-classical case	2
3	Weyl asymptotics for large eigenvalues	7

1 Introduction

PT-symmetry has been proposed as an alternative for self-adjointness in quantum physics [1, 2]. Thus for instance, if we consider a Schrödinger operator on \mathbf{R}^n ,

$$P = -h^2 \Delta + V(x), \quad (1.1)$$

*Ce travail a bénéficié d'une aide de l'Agence Nationale de la Recherche portant la référence ANR-08-BLAN-0228-01 ainsi que d'une bourse FABER du conseil régional de Bourgogne

the usual assumption of self-adjointness (implying that the potential V is real valued) can be replaced by that of PT-symmetry:

$$V \circ v = \overline{V}, \quad (1.2)$$

where $v : \mathbf{R}^n \rightarrow \mathbf{R}^n$ is an isometry with $v^2 = 1 \neq v$. If we introduce the parity operator $U_v u(x) = u(v(x))$ and the time reversal operator $\Gamma u = \overline{u}$, then this can be written

$$[P, U_v \Gamma] = 0. \quad (1.3)$$

Under mild additional technical assumptions it is easy to see that the spectrum of a PT-symmetric operator is invariant under reflexion in the real axis. However, in order to build PT-symmetric quantum physics it seems important that the spectrum be real, so a natural mathematical question is then to determine when so is the case. Results on reality and non-reality of the spectrum of PT-symmetric operators can be found in [12, 6, 7, 2].

The purpose of this note is to show that in a probabilistic sense “most” non-self-adjoint PT-symmetric operators that are symmetric in the sense of (2.4), have their eigenvalues distributed according to the Weyl law and hence many of their eigenvalues are non-real. As a matter of fact, this will be a rather easy adaptation of general results on the Weyl asymptotics for non-self-adjoint operators with small random perturbations [9, 10, 11, 4, 13, 14, 5], where the last three references are the ones that we shall use directly. For technical reasons we will state our results for elliptic operators on compact manifolds but it would be easy to adapt the results of [13] in order to treat Schrödinger operators on \mathbf{R}^n .

The addition of small random perturbations has the effect of destroying (uniform) analyticity (if the unperturbed operator has analytic coefficients). A very interesting question is to give criteria for PT symmetric operators with analytic coefficients to have real spectrum.

The plan of the paper is the following: In Section 2 we treat the semi-classical case and in Section 3 we treat the case of large eigenvalues.

2 The semi-classical case

Let X be a compact smooth manifold of dimension n . Let $v : X \rightarrow X$ be a smooth involution; $v^2 = \text{id}$, with $v \neq \text{id}$. Fix a smooth positive density dx on X which is invariant under v and let us take L^2 norms with respect to dx . Let P be a differential operator on X of order

$m \geq 2$ with smooth coefficients so that in local coordinates,

$$P = \sum_{|\alpha| \leq m} a_\alpha(x; h) (hD_x)^\alpha, \quad a_\alpha(\cdot; h) \in C^\infty. \quad (2.1)$$

Here $0 < h \ll 1$ is the semi-classical parameter and we assume that

$$a_\alpha(x; h) - a_\alpha(x; 0) = \mathcal{O}(h) \quad (2.2)$$

locally uniformly and similarly for all its derivatives. We also assume for simplicity that $a_\alpha(x; h) = a_\alpha(x)$ is independent of h when $|\alpha| = m$. Let

$$p(x, \xi) = \sum_{|\alpha| \leq m} a_\alpha(x; 0) \xi^\alpha, \quad p_m(x, \xi) = \sum_{|\alpha|=m} a_\alpha(x) \xi^\alpha$$

We assume that $p_m(x, \xi) \neq 0$ on $T^*X \setminus 0$, so that P is elliptic in the classical sense. We also assume that

$$p_m(T^*X) \neq \mathbf{C}. \quad (2.3)$$

Assume that P is symmetric,

$$P = \Gamma P^* \Gamma =: P^\dagger. \quad (2.4)$$

and that

$$PU = UP^*, \text{ where } Uu(x) = U_v u(x) := u(v(x)), \quad \Gamma u(x) = \overline{u(x)}. \quad (2.5)$$

This means that P is PT symmetric:

$$[U\Gamma, P] = 0. \quad (2.6)$$

In addition to the PT-symmetry property (2.6), we have assumed in (2.4) that P is symmetric.

Example 2.1 $P = -h^2 \Delta + V(x)$ on \mathbf{T}^n where $\Re V$ is even and $\Im V$ is odd, $V(-x) = \overline{V(x)}$. Then P is symmetric and PT-symmetric with $v(x) = -x$.

Let \tilde{R} be an auxiliary h -independent positive elliptic second order differential operator on X which commutes with U . We also assume that \tilde{R} is real, or equivalently that

$$[\Gamma, \tilde{R}] = 0. \quad (2.7)$$

Then \tilde{R} has an orthonormal basis of real eigenfunctions e_j such that $Ue_j = (-1)^{k(j)} e_j$ where $k(j) = 1$ or $k(j) = -1$. We say that e_j is even in the first case and odd in second case. Put $\epsilon_j = e_j$ when e_j is

even and $\epsilon_j = ie_j$ when e_j is odd. Then $\{\epsilon_j\}$ is also an orthonormal basis and a linear combination $V = \sum \alpha_j \epsilon_j$ is PT symmetric iff the coefficients α_j are real: $U(V) = \overline{V}$.

In order to formulate our result, we shall follow [14], where we treated a situation without any extra symmetry.

Let $\Omega \Subset \mathbf{C}$ be open, simply connected, not entirely contained in $\Sigma(p) := p(T^*X)$. Let $V_z(t) := \text{vol}(\{\rho \in \mathbf{R}^{2n}; |p(\rho) - z|^2 \leq t\})$. For $\kappa \in]0, 1]$, $z \in \Omega$, we consider the property that

$$V_z(t) = \mathcal{O}(t^\kappa), \quad 0 \leq t \ll 1. \quad (2.8)$$

Since $r \mapsto p(x, r\xi)$ is a polynomial of degree m in r with non-vanishing leading coefficient, we see that (2.8) holds with $\kappa = 1/(2m)$.

By $B_{\mathbf{R}^d}(0, r)$ we denote the open ball in \mathbf{R}^d with center 0 and radius r . Let q_ω be a random potential of the form,

$$q_\omega(x) = \sum_{0 < \mu_k \leq L} \alpha_k(\omega) \epsilon_k(x), \quad \alpha(\omega) = (\alpha_k(\omega))_{0 < \mu_k \leq L} \in B_{\mathbf{R}^D}(0, R), \quad (2.9)$$

where $\mu_k > 0$ are the square roots of the eigenvalues of $h^2 \widetilde{R}$ so that $h^2 \widetilde{R} \epsilon_k = \mu_k^2 \epsilon_k$. We choose $L = L(h)$, $R = R(h)$ in the interval

$$\begin{aligned} h^{\frac{\kappa-3n}{s-\frac{n}{2}-\epsilon}} \ll L \leq Ch^{-M}, \quad M \geq \frac{3n-\kappa}{s-\frac{n}{2}-\epsilon}, \\ \frac{1}{C} h^{-(\frac{n}{2}+\epsilon)M+\kappa-\frac{3n}{2}} \leq R \leq Ch^{-\widetilde{M}}, \quad \widetilde{M} \geq \frac{3n}{2} - \kappa + (\frac{n}{2} + \epsilon)M, \end{aligned} \quad (2.10)$$

for some $\epsilon \in]0, s - \frac{n}{2}[$, $s > \frac{n}{2}$, so by Weyl's law for the large eigenvalues of elliptic self-adjoint operators, the dimension D in (2.9) is of the order of magnitude $(L/h)^n$. We introduce the small parameter $\delta = \tau_0 h^{N_1+n}$, $0 < \tau_0 \leq \sqrt{h}$, where

$$N_1 := \widetilde{M} + sM + \frac{n}{2}. \quad (2.11)$$

The randomly perturbed PT symmetric operator is

$$P_\delta = P + \delta h^{N_1} q_\omega =: P + \delta Q_\omega. \quad (2.12)$$

Here (cf [13]) the exponent N_1 has been chosen so that we have uniformly for $h \ll 1$ and q_ω as above:

$$\|h^{N_1} q_\omega\|_{L^\infty} \leq \mathcal{O}(1) h^{-n/2} \|h^{N_1} q_\omega\|_{H_h^s} \leq \mathcal{O}(1),$$

where H_h^s is the natural semi-classical Sobolev space discussed in Section 2 of [14] with a norm equivalent to the standard norm in H^s for each fixed $h > 0$.

The random variables $\alpha_j(\omega)$ will have a joint probability distribution

$$P(d\alpha) = C(h)e^{\Phi(\alpha;h)}L(d\alpha), \quad (2.13)$$

where for some $N_4 > 0$,

$$|\nabla_\alpha \Phi| = \mathcal{O}(h^{-N_4}), \quad (2.14)$$

and $L(d\alpha)$ is the Lebesgue measure. ($C(h)$ is the normalizing constant, assuring that the probability of $B_{\mathbf{R}^D}(0, R)$ is equal to 1.)

We also need the parameter

$$\epsilon_0(h) = (h^\kappa + h^n \ln \frac{1}{h})(\ln \frac{1}{\tau_0} + (\ln \frac{1}{h})^2) \quad (2.15)$$

and assume that $\tau_0 = \tau_0(h)$ is not too small, so that $\epsilon_0(h)$ is small. Recall that $\Omega \Subset \mathbf{C}$ is open, simply connected, not entirely contained in $\Sigma(p)$. The main result of this section is:

Theorem 2.2 *Under the assumptions above, let $\Gamma \Subset \Omega$ have smooth boundary, let $\kappa \in]0, 1]$ be the parameter in (2.9), (2.10), (2.15) and assume that (2.8) holds uniformly for z in a neighborhood of $\partial\Gamma$. Then there exists a constant $C > 0$ such that for $C^{-1} \geq r > 0$, $\tilde{\epsilon} \geq C\epsilon_0(h)$ we have with probability*

$$\geq 1 - \frac{C\epsilon_0(h)}{r h^{n+\max(n(M+1), N_4+\widetilde{M})}} e^{-\frac{\tilde{\epsilon}}{C\epsilon_0(h)}} \quad (2.16)$$

that:

$$\begin{aligned} & |\#(\sigma(P_\delta) \cap \Gamma) - \frac{1}{(2\pi h)^n} \text{vol}(p^{-1}(\Gamma))| \leq \\ & \frac{C}{h^n} \left(\frac{\tilde{\epsilon}}{r} + C(r + \ln(\frac{1}{r}) \text{vol}(p^{-1}(\partial\Gamma + D(0, r)))) \right). \end{aligned} \quad (2.17)$$

Here $\#(\sigma(P_\delta) \cap \Gamma)$ denotes the number of eigenvalues of P_δ in Γ , counted with their algebraic multiplicity.

In the introduction of [13] there is a discussion about the choice of parameters which applies here also: Very roughly, if τ_0 is equivalent to some high power of h , then up to some power of $\ln(1/h)$, ϵ_0 is of the order of magnitude h^κ . Now choose $\tilde{\epsilon} = h^{\kappa-\kappa_0}$ for some $\kappa_0 \in]0, \kappa[$. When $\kappa > 1/2$, then the volume in (2.17) is $\mathcal{O}(r^\beta)$ with $\beta = 2\kappa - 1 > 0$ and more generally we may assume that it is $\mathcal{O}(r^\beta)$ for some $\beta > 0$. Then we choose r to be a suitable power of h and obtain that the right hand side in (2.17) is $\mathcal{O}(h^{\gamma-n})$ for some $\gamma > 0$. With these choices of the parameters we also see that the probability in (2.16) is very close to 1.

Proof of Theorem 2.2. We just have to make some small modifications in the proof of the main result in [14] (which in turn is a modification of the proof in [13]) and only mention the points where a difference appears. The proof in the two cited papers (see also the lecture notes [15]) uses three ingredients:

- 1) The construction of a special perturbation of the form δq_ω with q_ω as in (2.9) but with α in the *complex* ball $B_{\mathbf{C}^D}(0, R)$ for which we have nice lower bounds on the small singular values of P_δ in (2.12), see Proposition 7.3 in [13], Proposition 5.1 in [14].
- 2) A complex variable argument in the α variables using the existence of the special perturbation in step 1), which permits to conclude that we have nice lower bounds on a relative determinant for $P_\delta - z$, with probability close to 1.
- 3) Application of a proposition about the number of zeros of holomorphic functions with exponential growth. (See also [16] for an improved version of this proposition, not yet fully exploited.)

In the present situation we want our special perturbation $\delta q_\omega(x)$ to be PT-symmetric, that is we want the coefficients α in (2.9) to be real. All the parts of the proofs in step 1 immediately carry over to the case of real α except the following result which is the basic ingredient in the iterative process leading to the propositions mentioned above:

Let e_1, \dots, e_N be an ON family in $L^2(X)$ such that

$$\left\| \sum_1^N \lambda_j e_j \right\|_{H_h^s} \leq \mathcal{O}(1) \|\lambda\|_{\mathbf{C}^N}$$

where the constant $\mathcal{O}(1)$ is independent of the family and especially of N . Then there exists

$$q = \sum_{0 < \mu_j \leq L} \alpha_j e_j, \quad \alpha_j \in \mathbf{C}, \quad (2.18)$$

with $\|\alpha\|_{\mathbf{C}^D} \leq R$ with the parameters as in (2.10), such that

$$\|q\|_{H_h^s} \leq \mathcal{O}(1) h^{-\frac{n}{2}} N L^{s+\frac{n}{2}+\epsilon}$$

and such that the matrix

$$M_q = \left(\int q(x) e_j(x) e_k(x) dx \right)_{1 \leq j, k \leq N}$$

and its singular values

$$\|M_q\| = s_1(M_q) \geq \dots \geq s_N(M_q)$$

satisfy

$$\begin{aligned} \|M_q\| &\leq \mathcal{O}(1)Nh^{-n}, \\ s_k(M_q) &\geq h^n/\mathcal{O}(1), \text{ for } 1 \leq k \leq N/2. \end{aligned} \quad (2.19)$$

(See (6.23), (7.20), (7.23) in [13].)

Write $q = q_1 + iq_2$ where $q_1 = \sum (\Re \alpha_j) \epsilon_j$, $q_2 = \sum (\Im \alpha_j) \epsilon_j$, so that q_1 and q_2 are PT-symmetric. The upper bounds on $\|q\|_{H_h^s}$ and on $\|M_q\|$ follow from the bound $\|\alpha\| \leq R$ and therefore carry over to q_j . Since $M_q = M_{q_1} + iM_{q_2}$ we can apply the Ky Fan inequalities ([8]) and get

$$\frac{h^n}{\mathcal{O}(1)} \leq s_{2k-1}(M_q) \leq s_k(M_{q_1}) + s_k(M_{q_2}), \quad 1 \leq k \leq \frac{N}{4}.$$

Since the singular values are enumerated in decreasing order, it follows that for j equal to 1 or 2, we have

$$s_k(M_{q_j}) \geq \frac{h^n}{2\mathcal{O}(1)}, \quad 1 \leq k \leq \frac{N}{4}. \quad (2.20)$$

this means that step 1 can be carried out and we get a PT symmetric operator P_δ as in Proposition 5.1 in [14], the only slight difference is that rather than taking θ in $]0, 1/4[$ we have to confine this parameter to the smaller interval $]0, 1/8[$.

Step 2 now follows from Remark 8.3 in [13], where the main point is the reality of the coefficients α_j while the assumption of reality of the basis elements is not necessary, and was made there only because we had in mind a real perturbation.

Step 3 can be carried out without any modifications. \square

3 Weyl asymptotics for large eigenvalues

Let P^0 be an elliptic differential operator on X of order $m \geq 2$ with smooth coefficients and with principal symbol $p_m(x, \xi)$. In local coordinates we get, using standard multi-index notation,

$$P^0 = \sum_{|\alpha| \leq m} a_\alpha^0(x) D^\alpha, \quad p_m(x, \xi) = \sum_{|\alpha|=m} a_\alpha^0(x) \xi^\alpha. \quad (3.1)$$

Recall that the ellipticity of P^0 means that $p_m(x, \xi) \neq 0$ for $\xi \neq 0$. We assume that

$$p_m(T^*X) \neq \mathbf{C}. \quad (3.2)$$

As before we assume symmetry,

$$(P^0)^* = \Gamma P^0 \Gamma, \quad (3.3)$$

and that

$$P^0 U = U (P^0)^*, \quad (3.4)$$

with $U = U_v$ as in Section 2.

Let \tilde{R} be a reference operator as in and around (2.7) and define ϵ_j as there. Write

$$\tilde{R}\epsilon_j = (\mu_j^0)^2 \epsilon_j, \quad 0 < \mu_0^0 < \mu_1^0 \leq \mu_2^0 \leq \dots \quad (3.5)$$

so that $\mu_k = h\mu_k^0$ where μ_k are given after (2.9). Our randomly perturbed operator is

$$P_\omega^0 = P^0 + q_\omega^0(x), \quad (3.6)$$

where ω is the random parameter and

$$q_\omega^0(x) = \sum_0^\infty \alpha_j^0(\omega) \epsilon_j. \quad (3.7)$$

Here we assume that $\alpha_j^0(\omega)$ are independent real Gaussian random variables of variance σ_j^2 and mean value 0:

$$\alpha_j^0 \sim \mathcal{N}(0, \sigma_j^2), \quad (3.8)$$

where

$$(\mu_j^0)^{-\rho} e^{-(\mu_j^0)^{\frac{\beta}{M+1}}} \lesssim \sigma_j \lesssim (\mu_j^0)^{-\rho}, \quad (3.9)$$

$$M = \frac{3n - \frac{1}{2}}{s - \frac{n}{2} - \epsilon}, \quad 0 \leq \beta < \frac{1}{2}, \quad \rho > n, \quad (3.10)$$

where s, ρ, ϵ are fixed constants such that

$$\frac{n}{2} < s < \rho - \frac{n}{2}, \quad 0 < \epsilon < s - \frac{n}{2}.$$

Let $H^s(X)$ be the standard Sobolev space of order s . As we saw in [5] (where the random variables α_j^0 were complex valued), $q_\omega^0 \in H^s(X)$ almost surely since $s < \rho - \frac{n}{2}$. Hence $q_\omega^0 \in L^\infty$ almost surely, implying that P_ω^0 has purely discrete spectrum.

Consider the function $F(w) = \arg p_m(w)$ on S^*X . For given $\theta_0 \in S^1 \simeq \mathbf{R}/(2\pi\mathbf{Z})$, $N_0 \in \dot{\mathbf{N}} := \mathbf{N} \setminus \{0\}$, we introduce the property $P(\theta_0, N_0)$:

$$\sum_1^{N_0} |\nabla^k F(w)| \neq 0 \text{ on } \{w \in S^*X; F(w) = \theta_0\}. \quad (3.11)$$

Notice that if $P(\theta_0, N_0)$ holds, then $P(\theta, N_0)$ holds for all θ in some neighborhood of θ_0 . Also notice that if X is connected and X, p are analytic and the analytic function F is non constant, then $\exists N_0 \in \mathbf{N}$ such that $P(\theta_0, N_0)$ holds for all θ_0 .

We can now state the main result of this section, which is an adaptation of the main result of [5].

Theorem 3.1 *Assume that $m \geq 2$. Let $0 \leq \theta_1 \leq \theta_2 \leq 2\pi$ and assume that $P(\theta_1, N_0)$ and $P(\theta_2, N_0)$ hold for some $N_0 \in \mathbf{N}$. Let $g \in C^\infty([\theta_1, \theta_2];]0, \infty[)$ and put*

$$\Gamma_{\theta_1, \theta_2; 0, \lambda}^g = \{re^{i\theta}; \theta_1 \leq \theta \leq \theta_2, 0 \leq r \leq \lambda g(\theta)\}.$$

Then for every $\delta \in]0, \frac{1}{2} - \beta[$ there exists $C > 0$ such that almost surely: $\exists C(\omega) < \infty$ such that for all $\lambda \in [1, \infty[$:

$$\begin{aligned} & \left| \#(\sigma(P_\omega^0) \cap \Gamma_{\theta_1, \theta_2; 0, \lambda}^g) - \frac{1}{(2\pi)^n} \text{vol } p_m^{-1}(\Gamma_{\theta_1, \theta_2; 0, \lambda}^g) \right| \\ & \leq C(\omega) + C\lambda^{\frac{n}{m} - \frac{1}{m}(\frac{1}{2} - \beta - \delta)\frac{1}{N_0+1}}. \end{aligned} \quad (3.12)$$

The proof actually allows to have almost surely a simultaneous conclusion for a whole family of θ_1, θ_2, g :

Theorem 3.2 *Assume that $m \geq 2$. Let Θ be a compact subset of $[0, 2\pi]$. Let $N_0 \in \mathbf{N}$ and assume that $P(\theta, N_0)$ holds uniformly for $\theta \in \Theta$. Let \mathcal{G} be a subset of $\{(g, \theta_1, \theta_2); \theta_j \in \Theta, \theta_1 \leq \theta_2, g \in C^\infty([\theta_1, \theta_2];]0, \infty[)\}$ with the property that g and $1/g$ are uniformly bounded in $C^\infty([\theta_1, \theta_2];]0, \infty[)$ when (g, θ_1, θ_2) varies in \mathcal{G} . Then for every $\delta \in]0, \frac{1}{2} - \beta[$ there exists $C > 0$ such that almost surely: $\exists C(\omega) < \infty$ such that for all $\lambda \in [1, \infty[$ and all $(g, \theta_1, \theta_2) \in \mathcal{G}$, we have the estimate (3.12).*

The condition (3.9) allows us to choose σ_j decaying faster than any negative power of μ_j^0 . Then from the discussion below, it will follow that $q_\omega(x)$ is almost surely a smooth function. A rough and somewhat intuitive interpretation of Theorem 3.2 is then that for almost every PT symmetric elliptic operator of order ≥ 2 with smooth coefficients on a compact manifold which satisfies the conditions (3.2), (3.3), (3.4), the large eigenvalues distribute according to Weyl's law in sectors with limiting directions that satisfy a weak non-degeneracy condition.

Proof of Theorem 3.1. As already mentioned, the theorem is a variant of Theorem 1.1 in [5]. The difference is just that we now use real random variables in the perturbation q_ω^0 in order to assure the PT-symmetry while in [5] they were complex. The proof in [5] used a

reduction to the semi-classical case where the main result of [14] could be applied. The proof of Theorem 3.1 is an immediate modification of that proof, where we replace the main result in [14] by Theorem 2.2. The only point where the use of real Gaussian random variables instead of complex ones causes a slight change is the use of (4.10) in [5] that was established in [3], where we have to replace the denominator 2 by 4 in the case of real random variables. That was also proved by Bordeaux Montrieux in [3], Proposition 2.5.4. \square

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